STA 9715 - Applied Probability

Crowd-Sourced Final Exam Practice

The following problems have been suggested by the students of STA 9715 as helpful practice for the final exam. Note that the solutions shown here are from the students' submissions and have not been checked by the instructor.

Question:

A fair die is rolled twice, a biased coin with the probability of heads 0.7 is tossed three times, and a card is drawn from a standard deck of 52 playing cards. What is the probability that the die sum is greater than 6, the coin shows HTT, and the card drawn is a spade?

Solution:

Die: $\mathbb{P}(X > 6) = 7/12$

Coin: $\mathbb{P}(HTT) = (0.7) * (0.3) * (0.3) = 0.063$

Card: $\mathbb{P}(Spade) = 13/52 = 1/4 = 0.25$

Let A be the event that the sum of the die is greater than 6, the coin shows HTT, and the card drawn is a spade.

$\mathbb{P}(A) = (7/12) * (0.063) * (0.25) = 0.00918225$

Question History:

This is a spin on a couple of textbook questions, so I will say that it can be considered a "new" question. This question is intended to use the basic principles of counting probability and independence amongst the individual events, *i.e.*, coin toss and overall event A.

Instructor Notes:

I'd define three events (A, B, C) for the three outcomes of interest and then note that because they are independent, $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$.

Question:

Does the covariance between $\max(X, Y)$ and $\min(X, Y)$ equal the covariance between X and Y?

Solution:

No, since either the max is X and the min is Y or vice versa, and covariance is "symmetric" since the r.v. X does not equal the r.v. $\max(X, Y)$, nor does it equal the r.v. $\min(X, Y)$.

The following identities hold for X and Y:

$\max(X, Y) + \min(X, Y) = X + Y$, $\max(X, Y) * \min(X, Y) = XY$.

Using these, the covariance $\mathbb{C}[\max(X, Y), \min(X, Y)]$ can be expressed as:

$\mathbb{C}[\max(X, Y), \min(X, Y)] = \mathbb{E}[\max(X, Y)\min(X, Y)] - \mathbb{E}[\max(X, Y)]\mathbb{E}[\min(X, Y)]$

Substitute $\max(X, Y) * \min(X, Y) = XY$:

$\mathbb{C}[\max(X, Y), \min(X, Y)] = \mathbb{E}[XY] - \mathbb{E}[\max(X, Y)]\mathbb{E}[\min(X, Y)]$.

Compare this with $\mathbb{C}[X, Y]$:

$\mathbb{C}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$.

Question History:

This is a simplified version of BH 7.8.38 just asking the Covariance half of the problem in simplified terms just asking if the Covariance is the same between two random variables and the max and min of two random variables.

Instructor Notes:

To see that $\mathbb{E}[\max\{X, Y\}]\mathbb{E}[\min\{X, Y\}] \neq \mathbb{E}[X]\mathbb{E}[Y]$ in general, consider $X, Y \overset{\text{IID}}{\sim} \mathcal{N}(0, 1)$. Then, clearly, $\mathbb{E}[X]\mathbb{E}[Y] = 0 * 0 = 0$ but $\mathbb{E}[\max\{X, Y\}] > 0$ and $\mathbb{E}[\min\{X, Y\}] < 0$ so $\mathbb{E}[\max\{X, Y\}]\mathbb{E}[\min\{X, Y\}] < 0$.

Question:

Let $X_1, X_2, \ldots, X_3 \sim \text{Expo}(\lambda)$. The MGF of X_j is $\mathbb{M}_{X_j}(t) = \frac{\lambda}{\lambda - t}$.

a) Find the MGF of the sample mean $Y = \frac{1}{n}\sum_{j=1}^{n} X_j$

b) Use $\mathbb{M}_Y(t)$ to find the mean and variance of Y.

Solution:

a) $\mathbb{M}_Y(t) = \mathbb{M}_{X_1}(t/n)^n = \lambda^n/(\lambda - t/n)^n$

b) $\mathbb{M}'_Y(t) = \lambda^n/(\lambda - t/n)^{n+1}$ so $\mathbb{E}[Y] = \mathbb{M}'_Y(0) = \lambda^n/\lambda^{n+1} = 1/\lambda$

$\mathbb{M}''_Y(t) = (n+1)\lambda^n/n(\lambda - t/n)^{n+2}$ so $\mathbb{E}[Y^2] = \mathbb{M}''_Y(0) = \frac{n+1}{n\lambda^2}$.

this gives $\mathbb{V}[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = \frac{n+1}{n\lambda^2} - (1/\lambda)^2 = \frac{1}{n\lambda^2}$

Setting $n = 30$, we get $\mathbb{V}[Y] = 1/30\lambda^2$

Question History:

Originally BH 10.7-27, mofieid to use the exponential distribution instead of the Poisson distribution and including an additional question about using the properties of the MGF to compute the mean and variance

Question:

Let $\boldsymbol{Z}$ be a vector of 4 independent standard normal random variables. What is $\mathbb{E}[\|\boldsymbol{Z}\|^2]$?

Solution:

$\mathbb{E}[\|\boldsymbol{Z}\|^2] = 4$

Question History:

Modified from exam 2 question 14

Question History:

My practice problem for the 9715 Final is from Ch. 9.9 Q. 13 in Blitzstein Hwang Probability pdf (page 446). It is a conditional probability problem.

Problem:

A fair 4-sided die is rolled once. Find the expected number of additional rolls needed to obtain a value at least as large as that of the first roll.

Answer:

$E[Y] = \Sigma_{j=1}^{4} \frac{1}{j} = \frac{25}{12} = 2.083$

Original Question (Background + How-to-Solve):

A fair 6-sided die is rolled once. Find the expected number of additional rolls needed to obtain a value at least as large as that of the first roll.

Let X be the result of the first roll, so $X \sim \text{DiscreteUniform}(\{1, 2, 3, 4, 5, 6\})$. Additionally, let Y be the number of additional rolls necessary to get a roll $\geq X$. Then $\mathbb{P}[Y|X = k] = \frac{7-k}{6}$, where k = outcome of the first roll of the die, *i.e.*, X.

To see this,

$k \in \{1, 2, 3, 4, 5, 6\}$

7 represents the number above the max roll, or $6 + 1 = 7$

6 represents the total number of outcomes

Once the first roll (R_1) where $X = k$ is known, the probability of rolling a number at least as large as k on subsequent rolls is:

$P(R_1) = P(X = k) = \frac{1}{6}$

$P(R_2 \geq k) = \frac{7-k}{6}$

So for example, if $R_1 = X = k = 1$, then the probability that $R_2 \geq 1$ is $\frac{7-1}{6} = 1.00$ or 100%, because 1 is the minimum number on k's support.

If $R_1 = 3$, then $R_2 \geq 3 = \frac{7-3}{6} = \frac{4}{6} = \frac{2}{3} \approx 0.667 = 66.7\%$, which makes sense: for $R_2 \geq 3$,

$$R_2 = \begin{cases} 0 & Y \in \{1, 2\} \\ 1 & Y \in \{3, 4, 5, 6\} \end{cases}$$

We can have 4 true values and 2 false values. $\frac{4}{4+2} = \frac{4}{6} = \frac{2}{3}$

We need to find the Expected Value of Y to answer the question.

To find $E[Y]$, we need to find the expected value of Y at various points X.

Using the *Law of Total Expectation*:

$E[Y] = E[E[Y|X]]$

We established the conditional distribution of $Y|X = k \sim (\frac{7-k}{6})$ which is a finite distribution.

So the expected value of Y is the sum of the expected value of $Y|X = k \times P(X = k)$

We know the $P(X = k) = \frac{1}{6}$. Plugging it into the formula (and inverting the distribution for simplicity):

$\Sigma_{k=1}^{6} \frac{6}{7-k} \times \frac{1}{6} = \Sigma_{k=1}^{6} \frac{1}{7-k}$, replacing $\frac{1}{7-k}$ with some variable j:

$E[Y] = \Sigma_{k=1}^{6} \frac{1}{j} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = 2.45$

Instructor Notes:

By analogy with the original question, perhaps a bit clearer to write:

$$\sum_{i=1}^{4} \frac{1}{4} * \frac{1}{\frac{5-k}{4}} = \sum_{i=1}^{4} \frac{1}{4} * \frac{4}{5-k} = \sum_{i=1}^{4} \frac{1}{5-k} \approx 2.08333$$

Question:

A bakery produces batches of cookies where the weight of each cookie follows a normal distribution $\mathcal{N}(50, 10^2)$ grams. A batch contains 25 cookies. What is the probability that the mean weight of cookies in a batch is greater than 53 grams? Express your answer in terms of the standard normal CDF, $\Phi(z)$.

Solution:

Identify the sample mean and standard error:

Population mean: $\mu = 50$

Population standard deviation: $\sigma = 10$

Sample size: $n = 25$

Standard error (SE): $SE = \frac{\sigma}{\sqrt{n}} = \frac{10}{\sqrt{25}} = 2$

Set up the probability calculation:

$P(\bar{X} > 53) = P\left(Z > \frac{53-50}{2}\right)$

Standardize the sample mean:

$Z = \frac{\bar{X}-\mu}{SE} = \frac{53-50}{2} = 1.5$

Use the standard normal CDF:

$P(\bar{X} > 53) = 1 - \Phi(1.5)$

answer:

$P(\bar{X} > 53) = 1 - \Phi(1.5)$

Question History:

Central Limit Theorem Application

Question:

Let $X_1, X_2, ..., X_n$ be i.i.d. exponential random variables with mean λ^{-1}.

Derive the moment generating function (MGF) of X_i.

Solution:

For $X_i \sim \text{Exponential}(\lambda)$, the PDF is:

$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0$

The MGF is:

$\mathbb{M}_X(t) = \mathbb{E}[e^{tX}] = \int_0^\infty e^{tx} \lambda e^{-\lambda x} \, \mathrm{d}x$

Combine exponents:

$\mathbb{M}_X(t) = \int_0^\infty \lambda e^{-(\lambda - t)x} \, \mathrm{d}x$

For $t < \lambda$: $\mathbb{M}_X(t) = \frac{\lambda}{\lambda - t}$

Question History:

Moment Generating Function

A company has production lines, M and N. Line M produces 60% of the total products, while Line N produces 40%. From the quality checks:

- 5% of products from Line M are defective.
- 8% of products from Line N are defective.

If randomly selected product is found to be defective, what is the probability that it came from Line M.

Solution

To solve this, we use Bayes' Theorem.

Let M be an event that product came from Line M
Let N be an event that product came from Line N
Let D be the event that the product is defective

$$P(M|D) = \frac{P(D|M)\,P(M)}{P(D)}$$

$$P(D) = P(D|M)\,P(M) + P(D|N)\,P(N)$$
$$= (0.05 \times 0.6) + (0.08 \times 0.4)$$
$$= 0.03 + 0.032 = 0.062$$

$$P(M|D) = \frac{P(D|M)\,P(m)}{P(D)} = \frac{(0.05 \times 0.6)}{0.062} = \frac{0.03}{0.062} \approx 0.4839$$

Approximately, 48.39% of defective products came from Line M.

Question History: This is a Bayes' Theorem found on online probability tutorials and modified to include a real-world scenario, Especially the numbers.

Skills Tested

(i) Understanding Conditional probability.

(ii) Applying Bayes' Theorem

(iii) Calculating probabilities in real-world contexts.

Revised Homework Example:

Suppose a volatile stock can rise by 70%, drop by 50%, or remain unchanged on a given day, with probabilities p1, p2, and p3 respectively, where p1+p2+p3=1. The changes are independent from day to day. (a) Suppose a hedge fund manager always invests half of her current fortune into the stock each day. Let Yn be her fortune after n days, starting from an initial fortune of Y0=100. What happens to Yn as n→∞?

(b) More generally, suppose the hedge fund manager always invests a fraction α of her current fortune into the stock each day (in Part (a), we took α=1/2). With Y0 and Yn defined as in Part (a), find the function g(α) such that

$$\frac{\log Y_n}{n} \to g(\alpha)$$

with probability 1 as n→∞, and prove that g(α) is maximized for a certain value of α.

Part (a)

In this scenario, the hedge fund manager invests half of her current fortune into the stock each day. We want to determine the behavior of her fortune Yn as n→∞.

The expected multiplication factor for her fortune each day is given by:

$$E[M] = (1 + 0.7 \times 0.5)p_1 + (1 - 0.5 \times 0.5)p_2 + (1)p_3$$

Substituting in the values:

$$E[M] = (1 + 0.35)p_1 + (1 - 0.25)p_2 + 1p_3$$

$$E[M] = 1.35p_1 + 0.75p_2 + p_3$$

The expected log growth rate is:

$$\log(E[M]) = \log(1.35p_1 + 0.75p_2 + p_3)$$

Thus, as n→∞, the fortune Yn grows multiplicatively, and we have:

$$\frac{\log Y_n}{n} \to \log(E[M])$$

Part (b)

For the general case where the manager invests a fraction α of her fortune each day, we need to find the function g(α):

$$g(\alpha) = p_1 \log(1 + 0.7\alpha) + p_2 \log(1 - 0.5\alpha) + p_3 \log(1)$$

Simplifying, we have:

$$g(\alpha) = p_1 \log(1 + 0.7\alpha) + p_2 \log(1 - 0.5\alpha)$$

To find the value of α that maximizes g(α), we differentiate g(α) with respect to α and set the derivative equal to zero:

$$\frac{d}{d\alpha} g(\alpha) = 0.7p_1 \frac{1}{1 + 0.7\alpha} - 0.5p_2 \frac{1}{1 - 0.5\alpha} = 0$$

Solving this equation will provide the critical points. To determine which of these points maximizes g(α), we should examine the second derivative or analyze the behavior of g(α) at the endpoints.

Question History BH7.8: 88

This question has a similar topic to the turkey problem from the weekly quiz. using linearity of expectation, calculating variance formulas and probabilities with the normal CDF.

Question: A mother and a father have 8 children. The 10 heights in the family (in inches) are N(μ,σ2)r.v.s (with the same distribution, but not necessarily independent).

a.Assume for this part that the heights are all independent. On average, how many of the children are taller than both parents?

b.Let X1be the height of the mother, X2 be the height of the father, and Y1,…,Y8 be the heights of the children.Suppose that (X1,X2, Y1,…, Y8) is multivariate normal, with N(μ,σ2) marginals and Correlation: Corr(X1, Yj)=ρ for 1<=j<=8, with ρ<1. On average, how many of the children are more than 1 inch taller than their mother?

(1) $P(Y_j > X_1 \text{ and } Y_j > X_2)$

$P(Y_j > X_1) \cdot P(Y_j > X_2)$

$P(Y_j > X_1) = 0.5$. $P(Y_j > X_2) = 0.5$

$P(Y_j > X_1 \text{ and } Y_j > X_2) = 0.5 \times 0.5 = \frac{1}{4}$

$E[\text{number of children}] = 8 \cdot \frac{1}{4} = \boxed{2.}$

(2) we need find $P(Y_i > X_1 + 1)$

$D+1 = Y_i - X_1$

$E[D] = E[Y_i] - E[X_1] - 1 = \mu - \mu - 1 = -1$

$Var[D] = V(Y_i) + V(X_1) - 2 \cdot Cov(X_1, Y_j)$

$$\rho_{X_1 Y_j} = \frac{Cov(X_1, Y_j)}{\sqrt{V(X_1)V(Y_j)}} = Cov(X_1, Y_j) = \rho \cdot \sigma^2$$

$Var(D) = \sigma^2 + \sigma^2 - 2 \cdot \rho \cdot \sigma^2 = 2\sigma^2 - 2\rho\sigma^2 = 2\sigma^2(1-\rho)$

$D \sim N(-1, 2\sigma^2(1-\rho))$

$$P(D>0) = P\left(D > \frac{0-(-1)}{\sqrt{2\sigma^2(1-\rho)}}\right)$$

$$P(D>0) = 1 - \Phi\left(\frac{1}{\sqrt{2\sigma^2(1-\rho)}}\right)$$

$$\boxed{8 \cdot \left(1 - \Phi\left(\frac{1}{\sqrt{2\sigma^2(1-\rho)}}\right)\right)}$$

STA9715

1) Final exam practice question

Let assume $X_1, X_2, \ldots, X_{25}$ are i.i.d. random variables $\sim$ Exp with mean 2.

a) You can use Chebyshev's Inequality to calculate the probability that $\bar{X} = \frac{1}{25}\sum_{i=1}^{25} X_i$ deviates from the μ by more than 1.

b) Use Central Limit Theorem (CLT), derive an upper bound for the probability that $\bar{X} > 3$ using Chernoff's Inequality for Gaussian rv. Lets assume $\bar{X} \sim N(2, 0.16)$.

c) Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables from an exponential distribution with mean 2. Define $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i^2$. Use Law of Large Numbers and compute the value to which Y_n converges as $n \to \infty$

2) Solutions

a) By Chebyshev's Inequality

$\mu = 2$,

V[σ^2] $= 2^2 = 4$. V$\bar{X}$:

$\mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n} = \frac{4}{25} = 0.16$

Based on Chebyshev's Inequality for $|\bar{X} - \mu| > 1$:

$P(|\bar{X} - \mu| > 1) \leq \frac{\mathrm{Var}(\bar{X})}{1^2} = \frac{0.16}{1} = 0.16$

b) By (CLT)

$\bar{X} \sim N(\mu = 2, \sigma^2 = 0.16)$,

Chernoff - Gaussian $X \sim N(\mu, \sigma^2)$: $P(X > \mu + t) \leq \exp\left(-\frac{t^2}{2\sigma^2}\right)$, $\mu = 2, t = 3 - 2 = 1$, $\sigma^2 = 0.16$

$P(\bar{X} > 3) \leq \exp\left(-\frac{(1)^2}{2(0.16)}\right)$. $P(\bar{X} > 3) \leq \exp(-3.125)$.

The probability that $\bar{X} > 3$ is at most: $P(\bar{X} > 3) \leq 0.0435$.

c) Convergence of Y_n

$Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i^2$, where $X_i \sim \mathrm{Exp}(1/2)$:

$E[X_i^2] = \mathrm{Var}(X_i) + [E(X_i)]^2 = 4 + 4 = 8$

By(LLN):

$$Y_n \xrightarrow{P} 8 \quad \text{as } n \to \infty.$$

3) Question history

The question is a adaptation from Chapter 10, sugggested question from the syllabus (2,21,22) from the book BH Introduction to Probaility. The problem also include a exponential distribution. -Chebyshev's Inequality, to calculate the bounds propabilities -Central Limit Theorem, for normal approximation of sample means.- Chernoff Gaussian, Law of Large Numbers, for convergence of sample averages.

1) The practice question:

Example 2.6.

Three dice have the following probabilities of throwing a "six": p, q, r, respectively. One of the dice is chosen at random and thrown (each is equally likely to be chosen). A "six" appeared. What is the probability that the die chosen was the first one?

2) The solution to the practice question:

Solution:

The event "a 6 is thrown" is denoted by B and A_1,A_2 and A_3 denote the events that die 1, die 2 and die 3 was chosen. $P[A_1|B] = \dfrac{P[A_1 \cap B]}{P[B]} = \dfrac{P[B|A_1] \times P[A_1]}{P[B]} = \dfrac{p \times \frac{1}{3}}{P[B]}$. But

$$\begin{aligned} P[B] &= P[B \cap A_1] + P[B \cap A_2] + P[B \cap A_3] \\ &= P[B|A_1] \times P[A_1] + P[B|A_2] \times P[A_2] + P[B|A_3] \times P[A_3] \\ &= p \times \frac{1}{3} + q \times \frac{1}{3} + r \times \frac{1}{3} = \frac{p+q+r}{3} \\ &\Rightarrow \; P[A_1|B] = \frac{p \times \frac{1}{3}}{P[B]} = \frac{p \times \frac{1}{3}}{(p+q+r) \times \frac{1}{3}} = \frac{p}{p+q+r}. \end{aligned}$$

3) Question History:

ACTEX Learning, Study Manual for Exam P, 2nd Edition by Sam A. Broverman, Ph.D., ASA.

It is a question from the study manual in the section of Conditional Probability and Independence, I think it is suitable for our final exam practice.

EXAM QUESTION

Question 3) In a quality control test, n products are sampled, and the proportion of defective items is estimated as $\hat{p}_n = \frac{1}{n}\sum_{i=1}^{n} B_i$, where $B_i \sim \text{Bernoulli}(p)$. The company is now interested in the log-odds, defined as $\hat{L}_n = \log\left(\frac{\hat{p}_n}{1-\hat{p}_n}\right)$. Using CLT & delta method, approximate $var(\hat{L}_n)$.

HINT: First compute the asymptotic distribution of $\hat{p}_n$ using the CLT & then apply the Delta Method:
if $X \xrightarrow{d} N(\mu, \sigma^2)$, then $g(X) \xrightarrow{d} N(g(\mu), \sigma^2 \cdot g'(\mu)^2)$

SOLUTION:

① Asymptotic distribution (CLT):

$$\hat{p}_n \sim N\left(p, \frac{p(1-p)}{n}\right)$$

② Log-odds Transformation:

$$g(p) = \log\left(\frac{p}{1-p}\right) = \log(p) - \log(1-p)$$

$$g'(p) = \frac{1}{p} + \left(\frac{+1}{1-p}\right) = \frac{1}{p(1-p)}$$

③ Variance of $\hat{L}_n$ (Delta Method):

$$var(\hat{L}_n) \approx (g'(p))^2 * var(\hat{p}_n)$$

$$= \left(\frac{1}{p(1-p)}\right)^2 \times \frac{p(1-p)}{n}$$

$$= \frac{1}{np(1-p)}$$

$$\therefore \boxed{var(\hat{L}_n) = \frac{1}{np(1-p)}}$$

QUESTION HISTORY: Week 13 - Mini Quiz (Question 3)

Modification: This question now focuses on the log-odds function $g(p) = \log\left(\frac{p}{1-p}\right)$ instead of the odds ratio $g(p) = \frac{p}{1-p}$, changing the derivative & resulting variance calculation.

Extra Credit Assignment

Question 1)

A group of 120 voters must select a leader from 3 candidates: $A,\ B,\ C$. Each voter can cast one vote per cycle, there are four voting cycles in a day. The new leader is chosen if they receive at least 80 votes, $\frac{2}{3}$ of the total, in a single cycle.

Initially, all candidates are equally likely to be voted for $p_a vote = p_b vote = p_c vote = \frac{1}{3}$.

However, candidate A's favorability increases by 0.05 every 4 cycles (1 day). The probability of $B\ and\ C$ decrease equally to ensure the total probability sums to 1.

1) How many voting cycles will it take for A to win based on the expected number of votes per cycle.

2) Using the normal approximation method, verify the results from part one.

Hint: A's favorability increase can be represented by $p_a = \frac{1}{3} + 0.05k,\ where\ k = \#\ of\ days$

Answer part 1

$$P(\mu_a \geq 80)\ win\ condition$$

$$Votes\ for\ A \sim Binomial\ (n = 120,\ p = p_a vote) \rightarrow\ E[120 * p_a vote]$$

$$\mu_a = 120 p_a$$

Using the hint substitute p_a

$$\mu_a = 120\left(\frac{1}{3} + 0.05k\right) \geq 80$$

Solve for K

$$\frac{1}{3} + 0.05k \geq \frac{80}{120}$$

$$\frac{1}{3} + 0.05k \geq \frac{2}{3}$$

$$0.05k \geq \frac{1}{3}$$

$$k \geq 6.67$$

$k \approx 7\ days$ or approximately 28 voting cycles

Answer part 2 – using normal approximation

The number of votes for A in a single cycle has the following parameters:

$$\mu = 120p_a$$

Using the hint and answer from part 1 we can solve explicitly for p_a

$$p_a = \frac{1}{3} + 0.05(7) = 0.683$$

$$\mu_a = 120\left(\frac{1}{3} + 0.05(7)\right) = 81.96$$

$$\sigma = \sqrt{120p_a(1 - p_a)} = \sqrt{120 * 0.683(1 - 0.683)} \approx 5.36$$

For A to win $P(X_a \geq 80\ )$

Standard Normal $\rightarrow P(\mu + \sigma Z > 80)$

$$P(81.96 + 5.36Z > 80)$$

$$P(Z > -0.3656) = \Phi(-0.37) \approx 0.3557$$

Thus:

$$P(X_a \geq 80) = 1 - \Phi(-0.37) = 1 - 0.3557 = 0.6443$$

After 7 days, 28 voting cycles candidate A has over a 64% chance of winning $\frac{2}{3}$ of the vote.

Techniques used:

- Normal approximation to verify
- Expected Values

Inspiration: Conversation with my mom about a scene from a movie, Conclave where a new pope is elected, the process has been modified to make the question easier to answer.

From the BH textbook question 3.17 can be seen as somewhat similar.

Question 2)

The insurance claim for a random variable X is defined by the moment generating function.

$$M_x(t) = \left(1 - 350t\right)^{-4}$$

Find the first and second moment. Then find the standard deviation of the claim using the moment generating function below:

First moment

$$M_x'(t) = 1400\left(1 - 350t\right)^{-5}$$

$$M_x'(0) = 1400$$

$$E[X] = 1400$$

Second Moment

$$M_x''(t) = 2{,}450{,}000\left(1 - 350t\right)^{-6}$$

$$M_x''(0) = 2{,}450{,}000$$

$$Var[X] = E\left[X^2\right] - E[X]^2$$

$$Var[X] = 2{,}450{,}000 - (1400)^2$$

$$Var[X] = 490{,}000$$

$$\sigma = 700$$

Inspiration and sourcing:

Found and modified a practice question for moment generating functions. How to apply and use a Moment generating function to find the standard deviation of a problem.

Source: https://www.youtube.com/watch?v=gcpSImAQjlk

1) a) Sehun randomly chooses one of these ways to come to SMEnt. Travel time (T) of each way follow the normal distribution as follow:

- $T_1 \sim N(\mu_1; \sigma_1^2)$
- $T_2 \sim N(\mu_2; \sigma_2^2)$
- $T_3 \sim N(\mu_3; \sigma_3^2)$

Find $E(T)$ and $Var\,T$.

b) Find $E(T)$ and $Var(T)$ if travel time follows uniform distribution:

- $T_1 \sim U(a_1, b_1)$
- $T_2 \sim U(a_2, b_2)$
- $T_3 \sim U(a_3, b_3)$

c) Find $E(T)$ and $Var(T)$ if travel time follows binomial distribution:

- $T_1 \sim Bin(n_1, p_1)$
- $T_2 \sim Bin(n_2, p_2)$
- $T_3 \sim Bin(n_3, p_3)$

d) Find $E(T)$ & $Var(T)$ if travel time follows discrete distribution

- $T_1 \in (t_1; \dots t_{1k})$ w/ prob $P(T_1 = t_{1i}) = p_{1i}$
- $T_2 \in (t_2; \dots t_{2k})$ w/ prob $P(T_2 = t_{2i}) = p_{2i}$
- $T_3 \in (t_3; \dots t_{3k})$ w/ prob $P(T_3 = t_{3i}) = p_{3i}$

e) Find $E(T)$ and $Var(T)$ if travel time follows Poisson distribution

- $T_1 \sim Poisson(\lambda_1)$
- $T_2 \sim Poisson(\lambda_2)$
- $T_3 \sim Poisson(\lambda_3)$

~ Solution:

a) $E(T) = \frac{\mu_1 + \mu_2 + \mu_3}{3} = \bar{\mu}$

$V(T) = E[V(T|X)] + V[E(T|X)]$

$= \frac{\sigma_1^2 + \sigma_2^2 + \sigma_2^3}{3} + \frac{1}{3}[(\mu_1 - \bar{\mu})^2 + (\mu_2 - \bar{\mu})^2 + (\mu_3 - \bar{\mu})^2]$

b) $E(T_j) = \frac{a_j + b_j}{2} \Rightarrow E(T) = \frac{1}{3}\left[\frac{a_1 + b_1}{2} + \frac{a_2 + b_2}{2} + \frac{a_3 + b_3}{2}\right] = \bar{\mu}$

$V(T_j) = \frac{(b_j - a_j)^2}{12} \Rightarrow V(T) = \frac{1}{3}[V(T_1) + V(T_2) + V(T_3)] + V[E(T|X)]$

$= \frac{1}{3}[\sigma_1^2 + \sigma_2^2 + \sigma_2^3] + \frac{1}{3}[(\mu_1 - \bar{\mu})^2 + (\mu_2 - \bar{\mu})^2 + (\mu_3 - \bar{\mu})^2]$

c) $E(T_j) = n_j p_j \Rightarrow E(T) = \frac{n_1 p_1 + n_2 p_2 + n_3 p_3}{3} = \bar{\mu}$

$V(T_j) = n_j p_j (1 - p_j)$

$\Rightarrow V(T) = \frac{V(T_1) + V(T_2) + V(T_3)}{3} + V[E(T|X)]$

$= \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{3} + \frac{1}{3}[(\mu_1 - \bar{\mu})^2 + (\mu_2 - \bar{\mu})^2 + (\mu_3 - \bar{\mu})^2]$

d) $E(T_j) = \sum t_{ji} p_{ji}$

$\Rightarrow E(T) = \dfrac{E(T_1) + E(T_2) + E(T_3)}{3}$

$V(T_j) = \sum [(t_{ji} - E(T_j))] p_{ji}$

$\Rightarrow V(T) = \dfrac{Var(T_1) + V(T_2) + V(T_3)}{3} + [(\mu_1 - \bar{\mu})^2 + (\mu_2 - \bar{\mu})^2 + (\mu_3 - \bar{\mu})^2]$

2) $E(T_j) = \lambda_j \Rightarrow E(T) = \dfrac{\lambda_1 + \lambda_2 + \lambda_3}{3}$

$V(T_j) = \lambda_j \Rightarrow V(T) = \dfrac{\lambda_1 + \lambda_2 + \lambda_3}{3} + [(\mu_1 - \bar{\mu})^2 + (\mu_2 - \bar{\mu})^2 + (\mu_3 - \bar{\mu})^2]$

Question History : HW 9.1 modification

2) A person draws cards from a standard deck of 52 cards. The 1st card is revealed, and its rank is recorded. Find Expected value of additional draws required to draw another card with a rank at least

as high as first card

Solution:

X = rank of first card

Y = additional draws

- If first rank is k

$\Rightarrow P(X = k) = \frac{(14-k)4}{51}$

$\Rightarrow E(Y | X = k) = P(X=k) - 1 = \frac{51}{(14-k).4} - 1$

$\Rightarrow E(Y) = \sum_{k=1}^{13} E(Y | X = k) \, P(X = k)$

$= \sum_{k=1}^{13} \left[\frac{51}{(14-k)4} - 1 \right] \times \frac{4}{52}$

$= \frac{1}{13} \sum_{k=1}^{13} \frac{51}{(14-k)4} - \frac{1}{13} \sum_{k=1}^{13} 1$

$= \sum_{k=1}^{13} \frac{51}{4} \frac{1}{14-k} - 1$

$$= 13 \cdot \sum_{k=1}^{13} \frac{1}{14-k} - 1 = 13 \cdot \left(1 + \frac{1}{2} + \dots + \frac{1}{13}\right) - 1$$

$$\approx 2.1152$$

Question history : HW 9.13

3) Let X and Y be iid positive random variable c>0. Determine relationship of following expression

a) $E(X^2) \geq [E[X]]^2$ (Jensen's inequality)

b) $P(X > c) \leq \frac{E(X^k)}{c^k}$; $k > 1$ (Markov ineq)

c) $E(\sqrt{X}) \leq \sqrt{E(X)}$ ($\sqrt{x}$ concave, Jensen's inequal)

d) $E(e^Y) \geq e^{E(Y)}$ (e^x convex, Jensen)

f) $E[X \sin(Y)] =/? E(X) E[\sin(Y)]$
(= when X & sin Y independent; otherwise ?)

g) $V(X+Y) = V(X) + V(Y)$; X, Y indepedent

h) $E[\min(X, Y)] \leq \min[E(X), E(Y)]$
Jensen for concave func $\min(x, y)$

i) $P(X+Y > c) \geq P(X > c/2) + P(Y > c/2)$

⇒ lower bound for union probs

k) $Cov(X, Y) = 0$ (X ; Y inde)

l) $E(X^3 + Y^3) = E(X^3) + E(Y^3)$

linear of Expectation

m) $V(X^2) \neq [V(X)]^2$

V not behave quadratically

n) $P(X \leq Y) = / ?$ $P(X \geq Y)$

= when symmetric vars ; otherwise ?

o) $E[\max(X, Y)] \geq \max[E(X), E(Y)]$

Jensen ineq as max (x, y) is convex

p) $E(X^2 + Y^2) \leq [E(X) + E(Y)]^2$

Follow from $[E(X) + E(Y)]^2$

q) $Cov(X + Y, X - Y) = Var(X) - Var(Y)$

Covar definition

Question History : HW 10.7.

Compute the autocovariance & autocorrelation function of moving average m_t at lag = 1 ,where w_t is a white noise :

$$m_t = \frac{1}{6}w_{t-1} + \frac{2}{3}w_t + \frac{1}{6}w_{t+1},$$

Note: $Cov(w_t, w_{t+h}) = 0$

1. $Var(m_t) = Cov(\frac{1}{6}w_{t-1} + \frac{2}{3}w_t + \frac{1}{6}w_{t+1}, \frac{1}{6}w_{t-1} + \frac{2}{3}w_t + \frac{1}{6}w_{t+1}) =$

$$Cov(\frac{1}{6}w_{t-1}, \frac{1}{6}w_{t-1}) + Cov(\frac{1}{6}w_{t-1}, \frac{2}{3}w_t) + Cov(\frac{1}{6}w_{t-1}, \frac{1}{6}w_{t+1}) + Cov(\frac{2}{3}w_t, \frac{1}{6}w_{t-1}) + Cov(\frac{2}{3}w_t, \frac{2}{3}w_t)$$
$$+Cov(\frac{2}{3}w_t, \frac{1}{6}w_{t+1}) + Cov(\frac{1}{6}w_{t+1}, \frac{1}{6}w_{t-1}) + Cov(\frac{1}{6}w_{t+1}, \frac{2}{3}w_t) + Cov(\frac{1}{6}w_{t-1}, \frac{1}{6}w_{t+1})$$
$$\frac{1}{36}\sigma^2 + \frac{4}{9}\sigma^2 + \frac{1}{36}\sigma^2$$
$$\frac{1}{2}\sigma^2$$

2. $Cov(\frac{1}{6}w_{t-1} + \frac{2}{3}w_t + \frac{1}{6}w_{t+1}, \frac{1}{6}w_t + \frac{2}{3}w_{t+1} + \frac{1}{6}w_{t+2}) =$

$$Cov(\frac{1}{6}w_{t-1}, \frac{1}{6}w_t) + Cov(\frac{1}{6}w_{t-1}, \frac{2}{3}w_{t+1}) + Cov(\frac{1}{6}w_{t-1}, \frac{1}{6}w_{t+2}) + Cov(\frac{2}{3}w_t, \frac{1}{6}w_t) + Cov(\frac{2}{3}w_t, \frac{2}{3}w_{t+1})$$
$$+Cov(\frac{2}{3}w_t, \frac{1}{6}w_{t+2}) + Cov(\frac{1}{6}w_{t+1}, \frac{1}{6}w_t) + Cov(\frac{1}{6}w_{t+1}, \frac{2}{3}w_{t+1}) + Cov(\frac{1}{6}w_{t-1}, \frac{1}{6}w_{t+1})$$
$$\frac{1}{6}\frac{2}{3}\sigma^2 + \frac{2}{3}\frac{1}{6}\sigma^2$$
$$\frac{2}{9}\sigma^2$$

3. $$Corr = \frac{Cov(\frac{1}{6}w_{t-1} + \frac{2}{3}w_t + \frac{1}{6}w_{t+1}, \frac{1}{6}w_t + \frac{2}{3}w_{t+1} + \frac{1}{6}w_{t+2})}{(\sigma^4)^{1/2}} = \frac{\frac{2}{9}\sigma^2}{\frac{1}{2}\sigma^2} = \frac{4}{9}$$

This questions is a modification of the example in "Introduction to Time Series and Forecasting" Peter J. Brockwell, Richard A. Davis. Chapter 1 "Statinary Model and Autocorrelation Function"

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In [ ]:
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Final Exam Practice Questions (Crowdsourcing)

#1 The practice questions.

While Fred is sleeping one night, X legitimate emails and Y spam emails are sent to him. Suppose that X and Y are independent, with X ~ Pois(10) and Y ~ Pois(40). When he wakes up, he observes that he has 30 new emails in his inbox. Given this information, what is the expected value of how many new legitimate emails he has?

#2 The solution to your practice question.

The conditional distribution of X given X + Y = 30 is Bin(30, 10/50). So

$E(X|X + Y = 30) = 30 * 10/50 = 6.$

#3 The "Question History". This can be either i) a reference to a specific textbook problem and an explanation of how you modified it to be a suitable exam question; or ii) a claim that it is a new question and a description of what probability skill(s) it is intended to use.

Question from BH , chapter 9 Question no.2

Question:

A stick is broken into 3 pieces by picking 2 points uniformly along the stick and breaking the stick at those 2 points. What is the probability the 3 pieces can form a triangle?

Answer:

A triangle can be formed from (a, b, c) if and only if $a, b, c \in (0, 1/2)$. Let the two points of the break be x, y (WLOG assume $x < y$) so the three lengths are $x, y - x, 1 - y$. Plot these on a unit square and identify the regions where a triangle can be formed. Because the points are chosen uniformly at random, the probability of "triange-ability" is proportional to the area, here $1/8 + 1/8 = 1/4$

Question History:

BH 7.8.14(a). Designed to test use of uniform measure, joint distributions, and independence.

Question:

Suppose $Y_1, \ldots, Y_{25}$ are independent Gaussian random variables, with $Y_i \sim \mathcal{N}(10 + 0.1i, 4^2)$. Use the Chernoff bound and Chebyshev's inequality to find upper bounds on the probability that $\overline{Y}_{25} > 11.5$. Compare Chernoff's bound to Chebyshev's.

Solution:

Note that

$$\mu_{\overline{Y}_{25}} = \frac{1}{25}\sum_{i=1}^{25} \mu_i = \frac{1}{25}\sum_{i=1}^{25} = 11.3$$

and that the variance of the sum is $\sigma^2/n = 16/25 = (4/5)^2$.

To apply Chernoff, note

$$Z = \frac{\overline{Y}_{25} - 11.3}{4/5} = 0.25$$

so the Chernoff bound is $\mathbb{P}(Z > 0.25) < e^{-(0.25)^2/2} = e^{-0.03125}$.

To apply Chebyshev, note

$$\begin{aligned}
\mathbb{P}(\overline{Y}_{25} - \mu_{\overline{Y}_{25}} > 0.2) &= \mathbb{P}(\overline{Y}_{25} - 11.3 > 0.2) \\
&= \mathbb{P}(\overline{Y}_{25} > 11.5) \\
&\leq \frac{1}{(0.2/0.8)^2} \\
&= 16
\end{aligned}$$

The Chernoff bound is much tighter than the Chebyshev. We expect this since the Chebyshev makes far fewer assumptions.

Question History:

Week 13 Quiz and in class discussion comparing Chebysehv and Chernoff inequalities.

Instructor Notes:

I don't think there's a 'one-sided' Chebyshev to be used here. Additionally, both of these bounds are so loose as to be basically vacuous.

I think *conceptually* this question is right, but I would modify it to i) use two-sided deviations; ii) move further into the tails where the bounds are non-trivial.

Schrödinger has n boxes with cats, and vials of poison in them.

The cats have an equal probability of being alive or dead.

If we open each box in order one at a time, what is the expected number of boxes we need to open to get 2 living cats in a row?

Solution:

E(Wait until two living cats|first cat's alive) = 2 * (1/2) + (2 + E(Wait until two living cats)) * (1/2)

E(Wait until two living cats) = (2 * (1/2) + (2 + E(Wait until two living cats)) * (1/2)) * (1/2) + (1 + E(Wait until two living cats)) * (1/2)

E(WLL) = (1 + (1 + (1/2)*E(WLL)))*(1/2) + (1/2 + (1/2)*E(WLL)

E(WLL) = 1 + (1/4)*E(WLL) + 1/2 + (1/2)*E(WLL)

E(WLL) = 3/2 + (3/4)*E(WLL)

E(WLL) - (3/4)*E(WLL) = 3/2

(1/4)*E(WLL) = 3/2

E(WLL) = 6

Pulled from page 422 in the textbook, it's the heads tails problem.

Question:

Let X, Y be r.v.s such that $X \sim N(0,1)$. and conditional on $X = x$, $Y \sim N(ax, b^2)$, where $a > 0$, $b > 0$.

a) Find the joint PDF of X and Y

b) Are X and Y independent?

c) Compute the covariance between X and Y.

Solution:

a) $$f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y|x)$$

Since $X \sim N(0,1)$, $Y|X = x \sim N(ax, b^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi b^2}} e^{-\frac{(y-ax)^2}{2b^2}}$$

$$\Rightarrow f_{X,Y}(y,x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi b^2}} e^{-\frac{(y-ax)^2}{2b^2}}$$

$$= \frac{1}{2\pi b} e^{-\frac{x^2}{2}} \cdot e^{-\frac{(y-ax)^2}{2b^2}}$$

$$= \frac{1}{2\pi b} \exp\left(-\frac{x^2}{2} - \frac{(y-ax)^2}{2b^2}\right)$$

b) Since the conditional distribution of Y depends on X by the mean aX, and the value of X influences the expected value of Y which shows dependence, X and Y are not independent.

c) We know that $\text{Cov}(X,Y) = E[XY] - E[X]E[Y]$

Since $E[X] = 0$, $E[Y] = 0$. $X \sim N(0,1)$

$\Rightarrow \text{Cov}(X,Y) = E[X(aX)] = aE[X^2] = a \cdot 1 = a.$

Question History:

This question refers to BH. 7.11

Adapting from standard problems involving joint distributions and conditional expectations of normal r.v.s. It tests to compute the covariance between X and Y.

STA 9715
Final Exam Question
12/07/24

<u>Question</u>: Find the moment generating function of a Poisson distribution.

Hint: You may use the fact that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

Solution: Assume $X \sim Poisson(\lambda)$. The probability density function for a Poisson distribution is $P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$ for $k = 0, 1, 2, \ldots$. Thus,

$$E\left[e^{tX}\right] = \sum_{k=0}^{\infty} e^{tk} P\left(X = k\right)$$

$$= \sum_{k=0}^{\infty} e^{tk} \frac{\lambda^k e^{-\lambda}}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\left(e^t \lambda\right)^k}{k!}$$

$$= e^{-\lambda} e^{e^t \lambda}$$

$$= e^{\lambda\left(e^t - 1\right)}$$

Question history: This is a simplified version of BH 10.7 #28.